

# Critical non-equilibrium steady states of the Lorentz channel

A. K. Karlis,<sup>1,\*</sup> F. K. Diakonov,<sup>1,†</sup> C. Petri,<sup>2,‡</sup> and P. Schmelcher<sup>2,§</sup>

<sup>1</sup>*Department of Physics, University of Athens, GR-15771 Athens, Greece*

<sup>2</sup>*Zentrum für Optische Quantentechnologien, Universität Hamburg,  
Luruper Chaussee 149, 22761 Hamburg, Germany*

(Dated: January 4, 2012)

We investigate the transport properties of non-interacting particles propagating in a finite Lorentz channel (LC). We show that interparticle power-law correlations emerge, when the dynamics is described at a spatially coarse-grained level. This behaviour appears in the non-equilibrium steady state of the LC under flux boundary conditions and persists even in the presence of external driving, provided that the billiard's horizon is infinite in a static or temporal sense. We show that Fermi acceleration permits the synchronization of particle motion with the periodic appearance of the ballistic corridors, which, in turn, gives rise to intermittent dynamics and the emergence of critical correlations. Thus, for the driven setup, the critical state acts as an attractor possessing characteristics of self-organization.

PACS numbers: 05.60.Cd, 05.45.Ac, 05.45.Pq, 05.65.+b, 64.70.qj

The Lorentz gas (LG) [1] acts in the theory of dynamical systems as a paradigm allowing us to address fundamental issues of statistical mechanics, for instance, ergodicity and mixing [2–4], as well as transport processes, such as diffusion in the configuration space [5–9]. The static periodic LG comprises a regular lattice of circular fixed scatterers and an ensemble of non-interacting particles traveling freely between collisions and scattering elastically off the circular obstacles. The transport properties of such a system are determined by the billiard's geometry, that is the specific lattice symmetry and the lattice constant. If the maximum free path length is not bounded from above, then the setup possesses a so-called “infinite horizon” (IH) and the diffusion in configuration space is anomalous [10, 11]. For a more compact packing of the scatterers, i.e. “finite horizon” (FH), arbitrarily long flights are not possible and the system exhibits normal diffusion [6]. From a dynamical point of view, if the system has FH, then it is fully hyperbolic. However, in the case of the IH geometry, the chaoticity weakens and an ordered portion of phase space emerges, due to the existence of ballistic corridors, leading to an intermittent behaviour [12]. Independently of the specific geometry (FH or IH) it has also been rigorously proven that a finite LG under flux boundary conditions possesses a nontrivial nonequilibrium steady state [13].

Time-dependent generalizations of the original periodic Lorentz gas model have been introduced, in which the scatterers are allowed to oscillate [15–17], rendering the study of diffusion in momentum space possible. This process is intimately linked to Fermi acceleration [18], which is considered a fundamental acceleration mechanism in many areas of physics [19]. The mechanism consists in the increase of the mean energy of particles as a result of random collisions with moving scatterers.

In this Letter, we show for the first time the emergence of power-law (critical) correlations between the propa-

gating particles in the LG with IH in a channel geometry, despite the fact that they are non-interacting. To reveal these cross-correlations we introduce a spatially coarse-grained description of the dynamics. This critical behaviour appears in the nonequilibrium steady state established in the above setup, under flux boundary conditions. We demonstrate the existence of such a nonequilibrium steady state in particular in the presence of external driving and thereby, when the energy-shell is violated. We introduce the dynamically infinite horizon (DIH) as a property of driven extended billiards for which ballistic corridors – choosing appropriate geometrical and driving parameters – open up and close periodically in time, i.e. exist only for certain time intervals. In this case, we show that the development of Fermi acceleration enables the particles to synchronize their motion with the periodic appearance of the ballistic corridors, such that they can perform free flights of arbitrary length, which, in turn, gives rise to intermittent dynamics and the appearance of critical correlations. In this sense, it is shown that Fermi acceleration can act as an effective driving force to steer an ensemble of propagating particles towards a nonequilibrium steady state possessing several characteristics of self-organized criticality (SOC). This work adds to the recent intensive efforts to clarify the interrelation between intermittency, criticality and self-organization [20–23].

The channel is constructed by superimposing two square lattices of circular scatterers, which have the same lattice spacing  $w$ , and radii  $a$  and  $b$ , respectively. The lattices are positioned such that there is a  $b$  disk at the center of each unit cell of the lattice of  $a$  disks. The system is finite in both the  $x$  and  $y$  direction. In the  $x$  direction it extends from 0 to  $L$ , while two hard flat boundaries are placed at  $\pm w/2$ , confining the motion of the particles in the  $y$  direction. Particles are injected from the left end with a uniform distribution of direction angles in  $(-\pi/2, \pi/2)$  and a steady flow of 50 particles

per unit time. When reaching either end of the channel the particles escape the system. Hence, the considered device is a finite strip of a periodic LG, known as *Lorentz channel* (LC) [24–26]. Finally it is noted that the  $a$  disks are fixed in space, while the centers of the  $b$  disks can oscillate harmonically with amplitude  $A$ , as shown in Fig. 1. In the following, length is measured in units of  $w$  and time in units of  $1/\omega$ ,  $\omega$  being the angular frequency of the oscillation of the central disks. Without loss of generality, provided that the flat segments between the static discs are small enough to prohibit bouncing orbits in the  $y$  direction, we choose  $a = 0.48$  [14]. For the sake of brevity, in the following we analyze results obtained in the steady-state of the driven LC, i.e.  $A \neq 0$ . We remark that our findings with respect to the stationary properties of the steady state of the driven LC hold in a similar way for the static LC with IH [27].

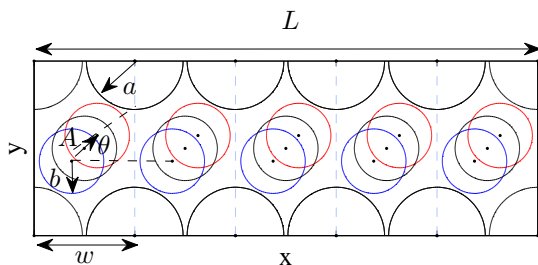


FIG. 1. (color online). Schematic of the billiard geometry. The central scatterers of radius  $b$  are driven harmonically, while the scatterers with radius  $a$  remain fixed.  $A$  and  $\theta$  are the amplitude and angle of oscillation, respectively. The oscillating disks are depicted in their equilibrium positions (black circles) together with the turning point positions (blue and red circles).

A derivation of the transport properties via the microscopic dynamics in such a system, even in the absence of driving, is impeded by the inherent hyperbolicity. However, statistical aspects of the transport properties of the infinite triangular static Lorentz gas, have been modeled by a suitable hopping process, either as a two-dimensional completely uncorrelated random walk [5] or by taking into account correlations between jumps [8, 28, 29]. Extending this description to the driven LG, and bearing in mind that the billiard under study is quasi-one-dimensional,  $L \gg 1$ , the transport properties can be studied by viewing the motion of the particles as a hopping process between neighboring cells [marked by dashed blue vertical lines in Fig. 1], i.e. as an one-dimensional stochastic process. This coarse-graining of the billiard, leads to a discrete chain consisting of  $N = L$  sites. At this level, the motion of a particle inside a unit cell – the intra-cell dynamics – is integrated out. There-

fore, we concentrate on the inter-cell properties of the motion of the particles.

Let us pose the question for the probability  $P(n)$  that a particle will travel  $n$  cells before changing its direction of the inter-cell motion. This probability is closely linked to the memory of the associated random walk process. To illustrate this, consider the time-independent LC, where the central disks are fixed at the equilibrium positions of the time-dependent setup. If the horizon is finite, i.e.  $\sqrt{3}/2 - a \leq b$ , then the system exhibits normal diffusion which can be quite accurately approximated by a simple random walk, with jump probability  $1/2$ . In this case,  $P(n)$  decays exponentially, i.e.  $P(n) \propto 2^{-n}$ . On the contrary, if ballistic corridors open up, the probability of long free flights between collisions is greatly enhanced and an algebraic decay of  $P(n)$  is expected [30, 31], reflecting the long-range correlations in the inter-cell dynamics.

Let us now focus on the driven LC. In Fig. 2(a) we present numerical results for  $P(n)$  obtained by the simulation of a LC with 100 cells with a representative incoming flux [32]. The radii of the central disks are  $b = \sqrt{3}/2 - a$ , which is the minimal value for the time-independent LC to possess a FH. The parameters of oscillation of the moving disks are chosen as follows: The amplitude is fixed to  $A = 1/2 - b$ , which is the maximum allowed value such that the disk motion is contained within the cell boundaries,  $\omega = 1$ , and we choose 5 different values for  $\theta$ . Evidently,  $P(n)$  shows a power-law tail for  $\theta \neq 0$ , which is the condition for the existence of dynamically open ballistic corridors. However, for small  $n$ ,  $P(n)$  is exponential, due to the dominance of diffusive transport. The value of  $n$  for which the crossover from exponential to power-law behaviour takes place depends on  $\theta$ , while the slope of the power-law tail is  $\theta$ -independent. For  $\theta = 0$  only an exponential decay is observed, since ballistic corridors remain closed for all times. Thus, for  $\theta \neq 0$  the particles synchronize with the oscillation of the central disks, such that long collisionless flights become possible. A detailed trajectory analysis verifies this and moreover, reveals that particles traveling long distances without changing their direction of motion from cell to cell demonstrate an intermittent behaviour. This is illustrated in Fig. 2(b)-(c), where the  $x$  coordinate of a particle and the corresponding velocity component are shown as a function of the coarse-grained time variable  $m$ , taking integer values. In fact  $m$  is increased by 1 either upon collision with the billiard's boundaries, or upon crossings across the border between neighboring cells. As observed, a trajectory exhibits phases of chaotic motion, where it stays localized, interrupted by regular (ballistic) motion. From a dynamical point of view, this corresponds to a transition from areas close to the ordered portions of phase space to the hyperbolic areas owing to the convex geometry of the billiard's boundaries. The regular part of the phase space exists in the high energy

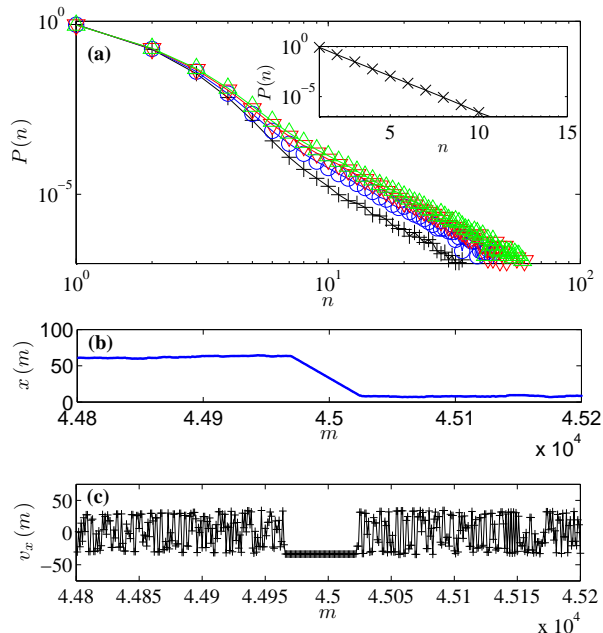


FIG. 2. (color online). (a) The probability of crossing  $n$  cells without changing direction of motion with  $b = \sqrt{3}/2 - a$ ,  $\omega = 1$  and  $|\mathbf{A}| = 1/2 - b$  and  $\theta = 0.11$  [upright black crosses],  $\theta = 0.22$  [blue open circles],  $\theta = 0.33$  [red downright triangles],  $\theta = 0.5$  [green upright triangles] and in the inset  $P(n)$  for  $\theta = 0$ , in units of  $\pi$ . (b)  $x$  coordinate; (c)  $v_x$  velocity of a typical trajectory versus the sum  $m$  of the number of collisions with the billiard's boundaries and crossings from cell to cell.

regime for certain combinations of the phase of oscillation of the moving disks and the direction of the velocities of the particles.

An interesting analogy can be drawn between the driven LC and the well-known sandpile model [33] (for a review see Ref. [34]). The redistribution of grains in the sandpile model, when out of equilibrium, can be described by a diffusion equation [35, 36]. However, when the sandpile reaches equilibrium the diffusion coefficient diverges and grains are redistributed with avalanches, the probability of the number of sites participating in such events being a power-law. In the driven LC the inter-cell motion of the particles can be described by a random walk, with finite moments of the distribution of step lengths, when the particle motion is not synchronized with the moving scatterers. As a result, long ballistic flights are not possible and a diffusion coefficient can be defined. As the non-equilibrium steady-state is reached, a subset of the ensemble of the particles propagating through the device will approach the areas of the phase space corresponding to ballistic motion. From that point on, particle dynamics will become intermittent and the simple random walk picture breaks down. The long flights without collisions—known as Lévy flights

in the framework of random walks—lead to the divergence of the diffusion coefficient, as it is the case in the time-independent LC with infinite horizon [31]. A manifestation of the Lévy type statistics is the appearance of the power-law tail of  $P(n)$ , similar to the appearance of “avalanches” in the sandpile SOC model. For the driven LC criticality implies the existence of ballistic corridors at least temporally which depends on the geometrical details of the set-up and the characteristics of the time-law of the driving. If ballistic corridors exist, then the critical state is an attractor, without the need of a fine-tuning of the external parameters.

To be able to study the robustness of the critical state under the variation of system parameters, we construct a complexity measure  $C$ , which is the product of two terms [37]: the first term is the quadratic distance of  $P(n)$  from the exponential behaviour expected for a fully hyperbolic system; the second term quantifies the amount of information—in the Shannon sense—stored in the system, not necessarily related to complex behaviour. To obtain a normalized complexity measure  $\tilde{C}$ ,  $C$  is divided with its maximum value obtained in the static, i.e. non-driven LC. By construction  $\tilde{C}$  is positive definite, maximizes when the probability of the occurrence of long collisionless flights becomes maximum and tends to zero when the motion of particles in the systems is completely chaotic [exponential  $P(n)$ ].

$$C = - \sum_{n=1}^N \left[ \log(P(n)) - \log\left(\frac{2^{-n}}{1 - 2^{-N}}\right) \right]^2 \times \sum_{n=1}^N \frac{P(n) \log(P(n))}{\log(N)}. \quad (1)$$

In Fig. 3, we present contour plots of the measure  $\tilde{C}$  as a function of the radii of the oscillating disks  $b$  and the angle of oscillation  $\theta$ . In Fig. 3(a), the oscillation of the central disks is synchronous, whereas in Fig. 3(b) the initial phases of the oscillating disks are chosen randomly according to a uniform distribution in the interval  $[0, 2\pi)$ . As seen in Fig. 3(a), the system always reaches the critical state, unless the parameters of oscillation and the radius  $b$  are such that ballistic corridors remain closed for all times. On the contrary, when the initial phases are randomly chosen dynamically open ballistic corridors cannot appear. Therefore, ballistic corridors exist only in the static sense, i.e. corridors remain continuously open despite the motion of the central disks. Thus, from geometrical considerations, the “phase boundary” is  $\sqrt{3} - 2a - (1 - 2b) \sin \theta - 2b = 0$ .

Given that the particles propagating through the LC are non-interacting, the question arises as to how the system can reach a critical state. The scale-free behaviour, as demonstrated by the power-law decay of  $P(n)$  at criticality, emerges at the level of the statistical ensemble because of the intermittency owing to the dynamically open

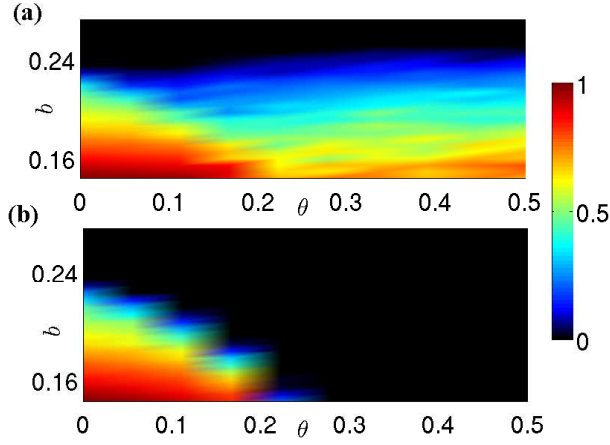


FIG. 3. (color online). A contour plot of  $\tilde{C}$  constructed by the simulation of the billiard's dynamics for 25 different values of  $b$  and 10 values of  $\theta$ . The black area corresponds to the non-critical state (exponential  $P(n)$ ). Panel (a) corresponds to synchronous oscillation of the central disks while (b) implies uniformly distributed random initial phases. Note that in both cases, as in Fig. 2,  $|\mathbf{A}| = 1/2 - b$ , which is the maximum allowed value of the oscillation amplitude  $|\mathbf{A}|$  for a given  $b$ .

ballistic corridors. These ordered structures deform the geometry of the phase space in such a way that correlations between independent trajectories, starting from different initial conditions, are induced. To verify this, we performed a symbolic dynamics analysis using a binary alphabet  $\mp 1$ , corresponding to a change or not of the direction of motion each time a particle crosses a cell. In Fig. 4 we present the correlation function computed on the basis of a long sequence, containing  $\sim 5 \times 10^6$  symbols, constructed by tracking a large number of trajectories. We observe that when the driven LC has a DIH (critical state) the correlation function  $G_{DIH}(n)$  decays algebraically. On the contrary, when the system is away from criticality (FH), the correlation function  $G_{FH}$  drops by several orders of magnitude in just a single step.

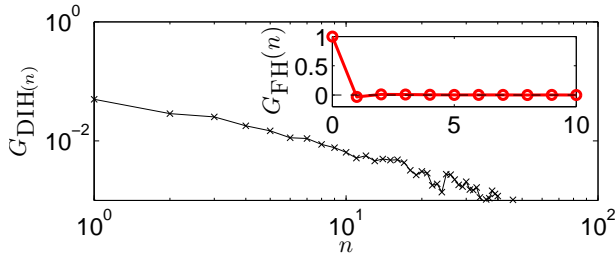


FIG. 4. (color online). The cross-correlation function  $G_{DIH}$  between independent trajectories in an DIH setup, with  $b = 0.16$ . In the inset  $G_{FH}(n)$  is also plotted for a set-up with FH  $b = a = 0.48$ .

In conclusion, we have shown that power-law (critical) correlations emerge between non-interacting parti-

cles propagating in the LC. This behaviour, revealed at a suitable spatial scale, is established in the nonequilibrium steady state reached in the static and driven LC. The key property leading to criticality is intermittency, which, combined with ergodicity, generates power-law cross-correlations. Intermittency implies the mixing of ordered with chaotic parts of trajectories at all time-scales, while ergodicity is responsible for transferring trajectory properties to the statistical ensemble. The interrelation of intermittent dynamics and critical behaviour has already been pointed out in Ref. [38], in the framework of the 3D Ising model. However, herein the mechanism leading to power-law correlations is reduced to its basic ingredients, due to the absence of any inter-particle interaction in the considered ensemble. This goes one step beyond the model of self-driven particles [39, 40], where inter-particle interactions are substituted by certain rules, leading to correlations. In this sense, this work sheds new light on the emergence of power-law correlations and criticality in dynamical systems in general. Finally, in the driven LC with DIH, we demonstrate a novel role of Fermi acceleration as an effective driving force steering an ensemble of propagating particles towards the critical state inducing this way characteristics of self-organization.

This work was made possible by the facilities of the Shared Hierarchical Academic Research Computing Network (SHARCNET:www.sharcnet.ca) and Compute/Calcul Canada. This research has been co-financed by the European Union (European Social Fund — ESF) and Greek national funds through the Operational Program "Education and Lifelong Learning" of the National Strategic Reference Framework (NSRF) - Research Funding Program: Heracleitus II. Investing in knowledge society through the European Social Fund. The authors also thank the IKY and DAAD for financial support in the framework of an exchange program between Greece and Germany (IKYDA).

\* karlis@physi.uni-heidelberg.de

† fdiakono@phys.uoa.gr

‡ cpetri@physnet.uni-hamburg.de

§ pschmelc@physnet.uni-hamburg.de

- [1] H. A. Lorentz, Proc. Roy. Acad., Amsterdam, 438, 585, 684, (1905).
- [2] Ya. G. Sinai, Russ. Math. Surveys **27**, 21 (1972); **25** 137 (1970).
- [3] L. A. Bunimovich, Comm. Math. Phys. **65**, 295 (1979).
- [4] L. A. Bunimovich and Ya. G. Sinai, Commun. Math. Phys. **78**, 247 (1980); **78**, 479 (1980).
- [5] J. Machta and R. Zwanzig, Phys. Rev. Lett. **50**, 1959 (1983).
- [6] H. v. Beijeren, Rev. Mod. Phys. **54**, 195 (1982).
- [7] P. Gaspard, *Chaos, Scattering, and Statistical Mechanics* (Cambridge University Press, Cambridge, 1998)

- [8] R. Klages and C. Dellago, J. Stat. Phys. **101**, 145 (2000).
- [9] D. N. Armstead, B. R. Hunt and E. Ott, Phys. Rev. E **67**, 021110 (2003).
- [10] P. M. Bleher, J. Stat. Phys. **66**, 315 (1992).
- [11] D. Szász and T. Varjú, J. Stat. Phys. **129**, 59 (2007).
- [12] R. Artuso and G. Cristadoro, J. Phys. A **37**, 85 (2004).
- [13] P. Gaspard, Physica A **240**, 54 (1997).
- [14] This would lead to intermittent dynamics within a unit cell, which is irrelevant for the subsequent discussion.
- [15] A. Yu. Loskutov, A. B. Ryabov, and L. G. Akinshin, J. Exp. Theor. Phys. **89**, 966 (1999); J. Phys. A **33** 7973 (2000).
- [16] A. K. Karlis, P. K. Papachristou, F. K. Diakonov, V. Constantoudis and P. Schmelcher, Phys. Rev. Lett. **97**, 194102 (2006); Phys. Rev. E **76** 016214 (2007).
- [17] D. F. M. Oliveira, J. Vollmer and E. D. Leonel, Physica D **240**, 389 (2011).
- [18] E. Fermi, Phys. Rev. **75**, 1169 (1949).
- [19] R. Blandford and D. Eichler, Phys. Rep. **154**, 1 (1987).
- [20] G. Y. Antar, S. I. Krasheninnikov, P. Devynck, R. P. Doerner, E. M. Hollmann, J. A. Boedo, S. C. Luckhardt, and R. W. Conn, Phys. Rev. Lett. **87**, 065001 (2001).
- [21] F. Sattin and M. Baiesi, Phys. Rev. Lett. **96**, 105005 (2006).
- [22] V. M. Uritsky, M. Paczuski, J. M. Davila and S. I. Jones, Phys. Rev. Lett. **99**, 025001 (2007).
- [23] B. A. Sun, H. B. Yu, W. Jiao, H. Y. Bai, D. Q. Zhao, and W. H. Wang Phys. Rev. Lett. **105**, 035501 (2010).
- [24] P. Gaspard, Chaos **3**, 427 (1993).
- [25] D. P. Sanders, Phys. Rev. E **71**, 016220 (2005).
- [26] D. Alonso, R. Artuso, G. Casati and I. Guarneri, Phys. Rev. Lett. **82** 1859 (1999).
- [27] A. K. Karlis, F. K. Diakonov and P. Schmelcher, in preparation.
- [28] R. Klages and N. Korabel, J. Phys. A **35**, 4823 (2002).
- [29] T. Gilbert and D. P. Sanders, Phys. Rev. E **80**, 041421 (2009).
- [30] J. P. Bouchaud and P. Le Doussal, J. Stat. Phys. **41**, 225 (1985).
- [31] H. Larralde, F. Leyvraz, G. Martinez-Mekler, R. Rechtman and S. Ruffo, Phys. Rev. E **58**, 4254 (1998).
- [32] For special fine-tuned initial conditions long free flights are possible, such as small intervals of angles and velocities. These are however nongeneric.
- [33] P. Bak, C. Tang and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987); Phys. Rev. A **38**, 364 (1988).
- [34] D. L. Turcotte, Rep. Prog. Phys. **62**, 1377 (1999).
- [35] L. P. Kadanoff, Phys. Today (March 1991), p. 9.
- [36] J. M. Carlson, J. T. Chayes, E. R. Grannan and G. H. Swindle, Phys. Rev. Lett. **65**, 2547 (1990).
- [37] R. López-Ruiz, H. L. Mancini and X. Calbet, Phys. Lett. A **209**, 321 (1995).
- [38] Y. F. Contoyiannis, F. K. Diakonov and A. Malakis, Phys. Rev. Lett. **89**, 035701 (2002).
- [39] T. Vicsek *et al.*, Phys. Rev. Lett. **75**, 1226 (1995).
- [40] C. Huepe and M. Aldana, Phys. Rev. Lett. **92**, 168701 (2004).